Engineering Calculus Formulas

Definition of the Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$$

Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

(Derivative of outside times the derivative of the inside.)

Derivatives of Logs and Exponentials

$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$
$\frac{d}{dx}(a^x) = a^x \ln a$	$\frac{d}{dx}(e^{ax}) = ae^{ax}$

Derivatives of Trig Functions

$\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$

Derivatives of Inverse Trig Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Derivatives of Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$
$\frac{d}{dx}(\cosh x) = \sinh x$
$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$
$\frac{d}{dx}(\csc x) = -\text{csch}\, x \coth$
$\frac{d}{dx}(\sec x) = -\text{sech}\, x \tanh x$
$\frac{d}{dx}(\cot x) = -\text{csch}^2 x$

Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$
$\frac{d}{dx}(\text{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
$\frac{d}{dx}(\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

Derivative of Inverse Functions

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Special Limits	$\lim_{x \to 0} (1 + x)^{1/x} = e$
$\lim_{x \to 0} \frac{\sin x}{x} = 1$	$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

When is a function not differentiable at a point?

a. f is not continuous at a
b. f has a corner at a
c. f has a vertical tangent at a

Definition of Continuity

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

Critical Point

An interior point c on the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist.

Engineering Calculus Formulas

Extreme values (absolute max/min) can occur at the end points of the interval of interest. End points are not relative max/min points.

Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there is at least one number c in $[a, b]$ such that $f(c) = k$

Precise Definition of a Limit

$$\lim_{x \to a} f(x) = L$$

If for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
If $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

Average Velocity

Is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$

$$\frac{f(a+\Delta t)-f(a)}{\Delta t}$$

Instantaneous Velocity

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t)-f(a)}{\Delta t} = f'(a)$$

Velocity, Speed, and Acceleration

Position at time t $\quad s = f(t)$

Velocity at time t $\quad v = \dfrac{ds}{dt} = f'(t)$

Speed at time t $\quad |v| = |f'(t)|$

Acceleration at time t $\quad a = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2} = f''(t)$

Average and Marginal Cost

Cost function $\quad C(x)$

Average cost $\quad \overline{C}(x) = \dfrac{C(x)}{x}$

Marginal cost $\quad C'(x)$ *(derivative)*

Linear Approximation of f at a

$L(x) = f(a) + f'(a)(x - a)$
$\Delta L = dy = f'(a)$
$\Delta x = f'(x)dx$

Differentials

Let f be differentiable on an interval containing x. A small change in x is denoted by the differential dx. The corresponding change in $y = f(x)$ is approximated by the differential
$\Delta y \approx f'(x) \, \Delta x$
$dy = xdx$
$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx$

Error $\dfrac{dy-\Delta y}{\Delta y} x 100$

Relative Error $\dfrac{\Delta F}{F} \approx \dfrac{dF}{F}$

Accuracy within a percentage
$|f(x) - L(x)| < \%$

Newton's Method for approx the zeros

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Make an initial estimate x_1 that is close to c. Determine the new approximation. (make a table)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If
$f(a) = f(b)$
Then there is at least one number c in (a, b) such that $f'(c) = 0$

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

L'Hopital's Rule

If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate forms:
$0/0, \infty/\infty, 0 \cdot \infty \; (rewrite \; f/g),$
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

L'Hopital's Rule Indeterminate form 1^0

$$\lim_{x \to \infty} e^{-x}\sqrt{x} = \frac{\sqrt{x}}{e^x} \text{, now } \frac{f(x)}{g(x)}$$

Engineering Calculus Formulas

Summation Formulas

1. $\sum\limits_{i=1}^{n} c = cn$

2. $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum\limits_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Riemann Sum of f on $[a, b]$ is $\sum\limits_{k=1}^{n} f(\overline{x}_k)\Delta x$

Left if $\overline{x}_k = a + (k-1)\Delta x$

Right if $\overline{x}_k = a + k\Delta x$

Midpoint if $\overline{x}_k = a + (k - \frac{1}{2})\Delta x$

For $k = 1, 2, ..., n$

Definite Integral

$$\int\limits_a^b f(x)dx = \lim\limits_{\Delta x \to 0} \sum\limits_{k=1}^{n} f(\overline{x}_k)\Delta x_k$$

Fundamental Theorem of Calculus (Part 1)

$A(x) = \int\limits_a^x f(t)dt$ continuous on [a,b] and

differentiable (a,b), $A'(x) = f(x)$

Fundamental Theorem of Calculus (Part 2)

$$\int\limits_a^b f(x)dx = F(b) - F(a)$$

Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int\limits_a^b f(x)\,dx$$

$$\int\limits_a^b f(x)dx = f(c)(b - a)$$

Displacement and Distance

Displacement of an object

$$s(b) - s(a) = \int\limits_a^b v(t)dt$$

Distance travelled $\int\limits_a^b |v(t)|dt$

Position from Velocity

$$s(t) = s(0) + \int\limits_0^t v(x)dx$$

Position from Acceleration

$$s(t) = s(0) + \int\limits_0^t a(x)dx$$

Trapezoidal Rule

Let f be continuous on the closed interval $[a, b]$.

$\int\limits_a^b f(x)dx \approx$

$\frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots$
$\hspace{4cm} + 2f(x_{n-1}) + f(x_n)]$

Simpson's Rule (n is even)

Let f be continuous on the closed interval $[a, b]$.

$\int\limits_a^b f(x)dx \approx$

$\frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$
$\hspace{4cm} + f(x_{n-1}) + f(x_n)]$

Limit involving e

$$\lim\limits_{x \to \infty}(1 + \frac{1}{x})^x = \lim\limits_{x \to \infty}(\frac{x+1}{x})^x = e$$

Area of a Region $A = \int\limits_a^b f(x)dx$

Area of a Region Between Two Curves

$$A = \int\limits_a^b (f(x) - g(x))\,dx$$

General Slicing Method

$$V = \int\limits_a^b A(x)\,dx$$

Perpendicular to x-axis means dx
Perpendicular to y-axis means dy

Disk Method about the x-axis

$$V = \int\limits_a^b \pi f(x)^2\,dx$$

Engineering Calculus Formulas

Washer Method about the x-axis

$$V = \int_a^b \pi(f(x)^2 - g(x)^2)\, dx$$

Volume by the Shell Method

$$V = \int_a^b 2\pi x(f(x) - g(x))\, dx$$

Arc Length

Let f have a continuous first derivative on the interval [a,b]

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Area of a Surface of Revolution

$$S = 2\pi \int_a^b r(x)\sqrt{1 + (f'(x))^2}\, dx$$

Work done by a variable force F in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\, dx$$

Solving Lifting Problems

The work required to lift the water is A(y) is the cross-sectional area of the horizontal slices and D(y) is the distance the slices must be lifted.

$$W = \int_a^b \rho g A(y)D(y)\, dy$$

Solving Force/Pressure Problems

w(y) width function. If the base is at y=0 and the top is at y=a, then the total force is

$$F = \int_0^a \rho g(a - y)w(y)\, dy$$

$a - y$ depth, $w(y)$ width

Exponential Decay Functions

$$y(t) = y_0 e^{-kt}$$

Half life is $T_{1/2} = \dfrac{ln2}{k}$

Absolute Error $= |c - x|$

Relative Error $= \dfrac{|c-x|}{|x|}$

Definition of the Limit of a Sequence

Let L be a real number. The limit of a sequence $\{a_n\}$ if L, written as $\lim\limits_{n \to \infty} a_n = L$

If for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n < M$. Sequences that have limits **converge**, whereas sequences that do not have limits **diverge**.

Taylor Polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 +$$
$$\cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

Maclaurin Polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 +$$
$$\cdots + \frac{f^{(n)}(0)}{n!}x^n$$

L'Hopital's Rule

If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate forms:

$0/0,\ \infty/\infty,\ 0 \cdot \infty\ (rewrite\ f/g),$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

L'Hopital's Rule Indeterminate form 1^0

$$\lim_{x \to \infty} e^{-x}\sqrt{x} = \frac{\sqrt{x}}{e^x}\ ,\ now\ \frac{f(x)}{g(x)}$$

Engineering Calculus Formulas

Test	Series	Converges	Diverges	Comment
nth-Term	$\sum\limits_{n=1}^{\infty} a_n$		$\lim\limits_{n \to \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum\limits_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum\limits_{n=1}^{\infty} \left(b_n - b_{n+1}\right)$	$\lim\limits_{n \to \infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim\limits_{n \to \infty} a_n = 0$		Remainder: $\|R_N\| \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum\limits_{n=1}^{\infty} a_n$ $a_n = f(n) \geq 0$	$\int\limits_{1}^{\infty} f(x)dx$ converges	$\int\limits_{1}^{\infty} f(x)dx$ diverges	Remainder: $0 < R_N < \int\limits_{N}^{\infty} f(x)dx$
Root	$\sum\limits_{n=1}^{\infty} a_n$	$\lim\limits_{n \to \infty} \sqrt[n]{\|a_n\|} < 1$	$\lim\limits_{n \to \infty} \sqrt[n]{\|a_n\|} > 1$	Test is inconclusive if $\lim\limits_{n \to \infty} \sqrt[n]{\|a_n\|} = 1.$
Ratio	$\sum\limits_{n=1}^{\infty} a_n$	$\lim\limits_{n \to \infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim\limits_{n \to \infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	Test is inconclusive if $\lim\limits_{n \to \infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$
Direct Comparison $(a_n, b_n > 0)$	$\sum\limits_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges	
Limit Comparison $(a_n, b_n > 0)$	$\sum\limits_{n=1}^{\infty} a_n$	$\lim\limits_{n \to \infty} \left(\frac{a_n}{b_n}\right) = L > 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges	$\lim\limits_{n \to \infty} \left(\frac{a_n}{b_n}\right) = L > 0$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges	

Limit involving e

$$\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim\limits_{x \to \infty} \left(\frac{x+1}{x}\right)^x = e$$

$ln(1 + x) \approx x$ (divergence test)